

Exactly solvable N -body quantum systems with $N = 3^k$ ($k \geq 2$) in the $D = 1$ dimensional space

A. Bachkhaznadj

Laboratoire de Physique Théorique, Département de Physique
Université Mentouri, Constantine, Algeria

M. Lassaut

Institut de Physique Nucléaire, CNRS-IN2P3, Université Paris-Sud,
Université Paris-Saclay, F-91406 Orsay Cedex, France

August 31, 2016

Abstract :

We study the exact solutions of a particular class of N confined particles of equal mass, with $N = 3^k$ ($k = 2, 3, \dots$), in the $D = 1$ dimensional space. The particles are clustered in clusters of 3 particles. The interactions involve a confining mean field, two-body Calogero type of potentials inside the cluster, interactions between the centres of mass of the clusters and finally a non-translationally invariant N -body potential. The case of 9 particles is exactly solved, in a first step, by providing the full eigensolutions and eigenenergies. Extending this procedure, the general case of N particles ($N = 3^k$, $k \geq 2$) is studied in a second step. The exact solutions are obtained via appropriate coordinate transformations and separation of variables. The eigenwave functions and the corresponding energy spectrum are provided.

PACS: 02.30.Hq, 03.65.-w, 03.65.Ge

1 Introduction

The study of exactly solvable integrable quantum systems of N interacting particles still retains attention. The Calogero [1, 2] and Sutherland [3, 4] models constitute famous examples. Most of the works have been performed in the $D = 1$ dimensional space. A good survey of the many-body problems can be found in [5, 6], and in the report [7], where many integrable quantum systems have been classified with respect to Lie algebras. Systems with point interactions have also been considered, still in $D = 1$ [8, 9]. The early works of Calogero and Sutherland have been extended to many more complex systems, concerning two- and three-body problems. We can quote, in a non exhaustive way, the works of several authors [10-19].

For four-body systems and beyond, the works on exactly solvable quantum systems are much more scarce [20-23]. Recently we have solved exactly some few-body quantum problems consisting of four, five and six particles moving on a line [24]. In these particular systems, the particles are confined in a harmonic trap and interact pairwise, in clusters of two and three particles, through two-body inverse square Calogero potentials. The obtained results suggest to extend the construction to larger systems of particles, even for N -body problems, with large values of N .

The purpose of the present paper is to point out a particular case of a N -body quantum problem admitting an exact solution. The interactions between the particles of this system are inspired by those used in our previous works for 3,4,5 and 6-particle systems [19, 23, 24]. The N particles, of equal masses, are confined in a harmonic mean field with frequency ω , and clustered in clusters of three particles. Then, the number N of particles is $N = 3^k$ ($k = 2, 3, \dots$). In each of 3^{k-1} three-body clusters, the particles interact mutually with two-body inverse square Calogero potentials [2]. Other many-particle interactions are added and chosen as follows : the clusters interact via a Calogero type of potential depending on the distance between their centres of mass. This can be generalized to $N = 3^k$ particles with $k \geq 2$ by clustering again the clusters in groups of 3 clusters, and letting the clusters of clusters to interact via their centres of mass. Finally a non-translationally invariant N -body interaction is added.

The hierarchy of clustering is illustrated in Fig.1 for the case of $N = 3^2$ particles. The case of 9 particles is studied and solved exactly. With some successive and appropriate coordinate transformations, the problem is solved by separation of variables. The solutions of the corresponding stationary Schrödinger equation are provided, namely the eigenwavefunctions and the corresponding eigenenergies. The general case of $N = 3^k$, ($k \geq 2$) is then studied by following the same procedure. After some straightforward calculations and coordinates transformations, the exact solutions are given.

The paper is organized as follows. In section 2 we present and solve the nine-body problem for the case of harmonic confinement of the particles. In section 3 we treat the case of $N = 3^k$ particles. Conclusions are given in section 4.

2 A nine-body problem with harmonic confinement

We consider the Hamiltonian :

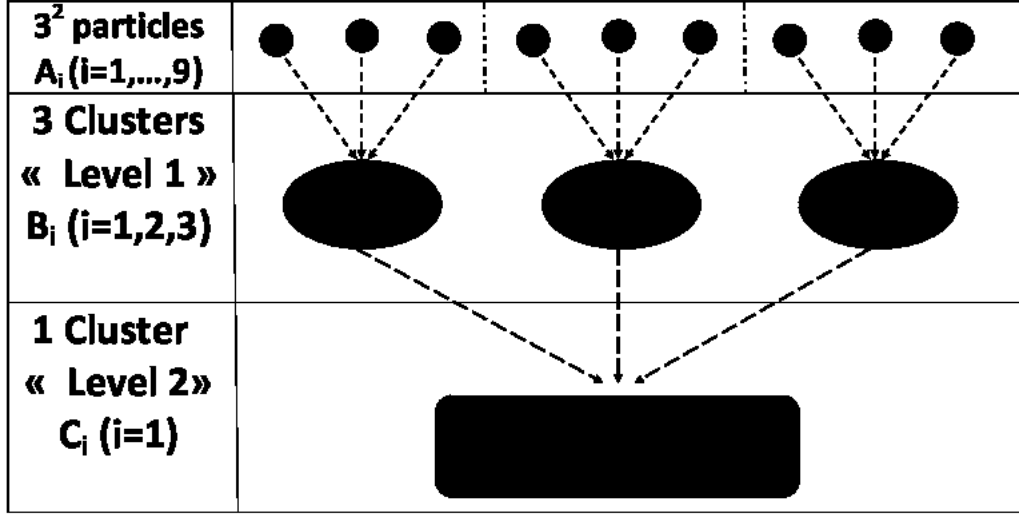


Figure 1: **A)** 3² particles in interaction, **B)** : 3 clusters of "level 1" each containing 3 particles, **C)** one cluster of "level 2" containing 3 clusters of "level 1".

$$\begin{aligned}
H = & \sum_{i=1}^9 \left(-\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,1}}{(x_i - x_j)^2} + \sum_{4 \leq i < j \leq 6} \frac{\lambda_{2,1}}{(x_i - x_j)^2} + \sum_{7 \leq i < j \leq 9} \frac{\lambda_{3,1}}{(x_i - x_j)^2} \\
& + \sum_{1 \leq i < j \leq 3} \frac{3\lambda_{1,2}}{(x_{3i-2} + x_{3i-1} + x_{3i} - x_{3j-2} - x_{3j-1} - x_{3j})^2} + \frac{\mu}{\sum_{i=1}^9 x_i^2} \quad (1)
\end{aligned}$$

Here, we use the units $\hbar = 2m = 1$. The first term gives the energy of the nine independent particles with coordinates $x_i, i = 1, 2, \dots, 9$ in a harmonic trap. The nine particles are clustered in 3 clusters of 3 particles. Inside each cluster, the particles interact pairwise via a two-body inverse square Calogero potential. The first cluster involves the first three particles, with coordinates x_1, x_2 and x_3 , the second one the next three particles, with coordinates x_4, x_5 and x_6 , and the third one the particles with coordinates x_7, x_8 and x_9 . The next-to-last terms represent the 3 clusters interacting pairwise via their centre of mass. It gives rise to the terms $(x_{3i-2} + x_{3i-1} + x_{3i} - x_{3j-2} - x_{3j-1} - x_{3j})^{-2}$ and constitutes actually a 6-body interaction. A non-translationally invariant nine-body potential with coupling constant μ is added, represented by the last term $\mu/(\sum_{i=1}^9 x_i^2)$.

In order to solve this nine-body problem, let us introduce the first coordinate transformation to Jacobi and centre of mass coordinates

$$u_{i,1} = \frac{1}{\sqrt{2}}(x_{3i-2} - x_{3i-1}), v_{i,1} = \frac{1}{\sqrt{6}}(x_{3i-2} + x_{3i-1} - 2x_{3i}) \quad (2)$$

$$w_{i,1} = \frac{1}{\sqrt{3}}(x_{3i-2} + x_{3i-1} + x_{3i}) \quad i = 1, 2, 3. \quad (3)$$

The second index "1" refers to this first coordinate transformation. The transformed Hamiltonian reads:

$$H = \sum_{i=1}^3 \left[-\frac{\partial^2}{\partial u_{i,1}^2} - \frac{\partial^2}{\partial v_{i,1}^2} - \frac{\partial^2}{\partial w_{i,1}^2} + \omega^2[u_{i,1}^2 + v_{i,1}^2 + w_{i,1}^2] + \frac{9\lambda_{i,1}[u_{i,1}^2 + v_{i,1}^2]^2}{2[u_{i,1}^3 - 3u_{i,1}v_{i,1}^2]^2} \right] + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,2}}{(w_{i,1} - w_{j,1})^2} + \frac{\mu}{\sum_{i=1}^3 [u_{i,1}^2 + v_{i,1}^2 + w_{i,1}^2]}. \quad (4)$$

This Hamiltonian is not separable in the 9 variables $\{u_{i,1}, v_{i,1}, w_{i,1}\}, i = 1, 2, 3$. We first introduce polar coordinates :

$$u_{i,1} = r_{i,1} \sin \varphi_{i,1}, \quad v_{i,1} = r_{i,1} \cos \varphi_{i,1}, \quad 0 \leq r_{i,1} < \infty, \quad 0 \leq \varphi_{i,1} \leq 2\pi, \quad i = 1, 2, 3. \quad (5)$$

By using the notations

$$\tilde{w}_1 \equiv \{w_{1,1}, w_{2,1}, w_{3,1}\}, \tilde{r}_1 \equiv \{r_{1,1}, r_{2,1}, r_{3,1}\}, \tilde{\varphi}_1 \equiv \{\varphi_{1,1}, \varphi_{2,1}, \varphi_{3,1}\}, \quad (6)$$

the Schrödinger equation is then written as :

$$\left\{ \sum_{i=1}^3 \left(-\frac{\partial^2}{\partial w_{i,1}^2} - \frac{\partial^2}{\partial r_{i,1}^2} - \frac{1}{r_{i,1}} \frac{\partial}{\partial r_{i,1}} + \omega^2[r_{i,1}^2 + w_{i,1}^2] \right) + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,2}}{(w_{i,1} - w_{j,1})^2} + \frac{\mu}{\sum_{i=1}^3 (r_{i,1}^2 + w_{i,1}^2)} + \sum_{i=1}^3 \frac{1}{r_{i,1}^2} \left[-\frac{\partial^2}{\partial \varphi_{i,1}^2} + \frac{9\lambda_{i,1}}{2 \sin^2(3\varphi_{i,1})} \right] - E \right\} \Psi(\tilde{w}_1, \tilde{r}_1, \tilde{\varphi}_1) = 0. \quad (7)$$

The potential involved in the equation (7)

$$V(\tilde{w}_1, \tilde{r}_1, \tilde{\varphi}_1) = \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,2}}{(w_{i,1} - w_{j,1})^2} + \frac{\mu}{\sum_{i=1}^3 (r_{i,1}^2 + w_{i,1}^2)} + \sum_{i=1}^3 \left[\frac{1}{r_{i,1}^2} \frac{9\lambda_{i,1}}{2 \sin^2(3\varphi_{i,1})} \right]$$

has the general form

$$V(\tilde{w}_1, \tilde{r}_1, \tilde{\varphi}_1) = f(\tilde{w}_1, \tilde{r}_1) + \sum_{i=1}^3 \frac{f_i(\varphi_{i,1})}{r_{i,1}^2}. \quad (8)$$

This suggests the wave function to be factorized as follows

$$\Psi(\tilde{w}_1, \tilde{r}_1, \tilde{\varphi}_1) = \chi(\tilde{w}_1, \tilde{r}_1) \times \prod_{i=1}^3 \Phi_{i,1}(\varphi_{i,1}). \quad (9)$$

The equation (7) will be solved in two steps. Firstly we consider the 3 angular equations :

$$\left(-\frac{d^2}{d\varphi_{i,1}^2} + \frac{9\lambda_{i,1}}{2\sin^2(3\varphi_{i,1})}\right)\Phi_{n_{i,1}}(\varphi_{i,1}) = B_{n_{i,1}}\Phi_{n_{i,1}}(\varphi_{i,1}), \quad i = 1, 2, 3, \quad (10)$$

on the interval $]0, \pi/3[$, with Dirichlet conditions at the boundaries. In the vicinity of $\varphi_{i,1} = 0$, ($i = 1, 2, 3$) (resp. $\frac{\pi}{3}$), the singularity can be treated if and only if $\lambda_{i,1} > -1/2$, similar to the case of a centrifugal barrier. Otherwise the operator has several self-adjoint extensions, each of which may lead to a different spectrum [25, 26].

The $B_{n_{i,1}}$, $i = 1, 2, 3$, are the eigenvalues of the equations (10), respectively given by [10, 19]

$$B_{n_{i,1}} = b_{n_{i,1}}^2, \quad b_{n_{i,1}} = 3\left(n_{i,1} + \frac{1}{2} + a_{i,1}\right), \quad (11)$$

$$a_{i,1} = \frac{1}{2}\sqrt{1 + 2\lambda_{i,1}}, \quad \left(\lambda_{i,1} > -\frac{1}{2}\right), \quad n_{i,1} = 0, 1, 2, \dots, \quad i = 1, 2, 3. \quad (12)$$

The associated eigensolutions are given in terms of the Gegenbauer polynomials $C_n^{(q)}$ [27]

$$\begin{aligned} \Phi_{n_{i,1}}(\varphi_{i,1}) &= (\sin 3\varphi_{i,1})^{\frac{1}{2}+a_{i,1}} C_{n_{i,1}}^{(\frac{1}{2}+a_{i,1})}(\cos 3\varphi_{i,1}), \\ 0 \leq \varphi_{i,1} &\leq \frac{\pi}{3}, \quad n_{i,1} = 0, 1, 2, \dots \end{aligned} \quad (13)$$

The extension from the interval $]0, \pi/3[$ to the whole interval $[0, 2\pi]$ is made following the prescription given in [10] by using symmetry arguments according to the statistics obeyed by the particles.

The second step consists in the resolution of the following Schrödinger equation :

$$\begin{aligned} &\left\{ \sum_{i=1}^3 \left[-\frac{\partial^2}{\partial w_{i,1}^2} - \frac{\partial^2}{\partial r_{i,1}^2} - \frac{1}{r_{i,1}} \frac{\partial}{\partial r_{i,1}} + \omega^2[r_{i,1}^2 + w_{i,1}^2] \right] + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,2}}{(w_{i,1} - w_{j,1})^2} \right. \\ &\quad \left. + \frac{\mu}{\sum_{i=1}^3 [r_{i,1}^2 + w_{i,1}^2]} + \sum_{i=1}^3 \frac{B_{n_{i,1}}}{r_{i,1}^2} - E_{\tilde{n}_1} \right\} \chi_{\tilde{n}_1}(\tilde{w}_1, \tilde{r}_1) = 0, \end{aligned} \quad (14)$$

with $\tilde{n}_1 \equiv \{n_{1,1}, n_{2,1}, n_{3,1}\}$.

We first draw the attention to $w_{i,1}$'s variables and introduce the coordinates labeled by the second index "2", corresponding to the second set of the coordinates transformation to the Jacobi and centre of mass coordinates

$$u_{1,2} = \frac{1}{\sqrt{2}}(w_{1,1} - w_{2,1}), v_{1,2} = \frac{1}{\sqrt{6}}(w_{1,1} + w_{2,1} - 2w_{3,1}), w_{1,2} = \frac{1}{\sqrt{3}}(w_{1,1} + w_{2,1} + w_{3,1}). \quad (15)$$

The transformed equation reads:

$$\left\{ -\frac{\partial^2}{\partial u_{1,2}^2} - \frac{\partial^2}{\partial v_{1,2}^2} - \frac{\partial^2}{\partial w_{1,2}^2} - \sum_{i=1}^3 \left(\frac{\partial^2}{\partial r_{i,1}^2} + \frac{1}{r_{i,1}} \frac{\partial}{\partial r_{i,1}} \right) + \sum_{i=1}^3 \frac{B_{n_{i,1}}}{r_{i,1}^2} \right\}$$

$$\begin{aligned}
& +\omega^2[u_{1,2}^2 + v_{1,2}^2 + w_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2] + \frac{\mu}{u_{1,2}^2 + v_{1,2}^2 + w_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2} \\
& + \frac{9\lambda_{1,2}[u_{1,2}^2 + v_{1,2}^2]^2}{2[u_{1,2}^3 - 3u_{1,2}v_{1,2}^2]^2} - E_{\tilde{n}_1} \Big\} \chi_{\tilde{n}_1}(u_{1,2}, v_{1,2}, w_{1,2}, \tilde{r}_1) = 0 . \quad (16)
\end{aligned}$$

We then introduce the following transformation to polar coordinates :

$$u_{1,2} = r_{1,2} \sin \varphi_{1,2}, \quad v_{1,2} = r_{1,2} \cos \varphi_{1,2}, \quad 0 \leq r_{1,2} < \infty, \quad 0 \leq \varphi_{1,2} \leq 2\pi . \quad (17)$$

The Schrödinger equation(16) becomes :

$$\begin{aligned}
& \left\{ -\frac{\partial^2}{\partial w_{1,2}^2} - \frac{\partial^2}{\partial r_{1,2}^2} - \frac{1}{r_{1,2}} \frac{\partial}{\partial r_{1,2}} - \sum_{i=1}^3 \left(\frac{\partial^2}{\partial r_{i,1}^2} + \frac{1}{r_{i,1}} \frac{\partial}{\partial r_{i,1}} \right) + \omega^2[w_{1,2}^2 + r_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2] \right. \\
& + \frac{\mu}{w_{1,2}^2 + r_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2} + \frac{1}{r_{1,2}^2} \left[-\frac{\partial^2}{\partial \varphi_{1,2}^2} + \frac{9\lambda_{1,2}}{2 \sin^2(3\varphi_{1,2})} \right] \\
& \left. + \sum_{i=1}^3 \frac{B_{n_{i,1}}}{r_{i,1}^2} - E_{\tilde{n}_1} \right\} \chi_{\tilde{n}_1}(w_{1,2}, r_{1,2}, \varphi_{1,2}, \tilde{r}_1) = 0 . \quad (18)
\end{aligned}$$

The potential involved in the equation (18)

$$\begin{aligned}
V(w_{1,2}, r_{1,2}, \varphi_{1,2}, \tilde{r}_1) &= \omega^2[w_{1,2}^2 + r_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2] + \frac{\mu}{w_{1,2}^2 + r_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2} \\
&+ \sum_{i=1}^3 \frac{B_{n_{i,1}}}{r_{i,1}^2} + \frac{1}{r_{1,2}^2} \frac{9\lambda_{1,2}}{2 \sin^2(3\varphi_{1,2})} , \quad (19)
\end{aligned}$$

has the general form

$$V(w_{1,2}, r_{1,2}, \varphi_{1,2}, \tilde{r}_1) = f_1(w_{1,2}, r_{1,2}, \tilde{r}_1) + \frac{f(\varphi_{1,2})}{r_{1,2}^2} . \quad (20)$$

Again, it suggests the wave function $\chi_{\tilde{n}_1}$ to be factorized as follows :

$$\chi_{\tilde{n}_1}(w_{1,2}, r_{1,2}, \varphi_{1,2}, \tilde{r}_1) = \frac{1}{\sqrt{r_{1,2} \prod_{i=1}^3 r_{i,1}}} \eta_{\tilde{n}_1}(w_{1,2}, r_{1,2}, \tilde{r}_1) \Phi_{1,2}(\varphi_{1,2}) . \quad (21)$$

The equation (18) will be solved in two steps. Firstly we solve

$$\left(-\frac{d^2}{d\varphi_{1,2}^2} + \frac{9\lambda_{1,2}}{2 \sin^2(3\varphi_{1,2})} \right) \Phi_{n_{1,2}}(\varphi_{1,2}) = B_{n_{1,2}} \Phi_{n_{1,2}}(\varphi_{1,2}), \quad (22)$$

on the interval $]0, \pi/3[$, with Dirichlet conditions at the boundaries. Note that the condition $\lambda_{1,2} > -\frac{1}{2}$ ensures that the operator is self-adjoint. The latter equation shows that $\Phi_{n_{1,2}}$ does not depend on the index \tilde{n}_1 .

$B_{n_{1,2}}$ denotes the eigenvalues of Eq.(22), given by

$$B_{n_{1,2}} = b_{n_{1,2}}^2 \quad b_{n_{1,2}} = 3 \left(n_{1,2} + \frac{1}{2} + a_{1,2} \right), \quad (23)$$

$$a_{1,2} = \frac{1}{2} \sqrt{1 + 2\lambda_{1,2}} \quad \left(\lambda_{1,2} > -\frac{1}{2} \right), \quad n_{1,2} = 0, 1, 2, \dots \quad (24)$$

The associated eigensolutions are written in terms of the Gegenbauer polynomials $C_M^{(q)}$

$$\begin{aligned} \Phi_{n_{1,2}}(\varphi_{1,2}) &= [\sin 3(\varphi_{1,2})]^{\frac{1}{2}+a_{1,2}} C_{n_{1,2}}^{(\frac{1}{2}+a_{1,2})}(\cos 3\varphi_{1,2}), \\ 0 &\leq \varphi_{1,2} \leq \frac{\pi}{3}, \quad n_{1,2} = 0, 1, 2, \dots \end{aligned} \quad (25)$$

Then, we have to solve the following Schrödinger equation

$$\begin{aligned} &\left\{ -\frac{\partial^2}{\partial w_{1,2}^2} - \frac{\partial^2}{\partial r_{1,2}^2} - \sum_{i=1}^3 \left[\frac{\partial^2}{\partial r_{i,1}^2} \right] + \omega^2 [r_{1,2}^2 + w_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2] + \frac{\mu}{r_{1,2}^2 + w_{1,2}^2 + \sum_{i=1}^3 r_{i,1}^2} \right. \\ &\left. + \sum_{i=1}^3 \frac{B_{n_{i,1}} - \frac{1}{4}}{r_{i,1}^2} + \frac{B_{n_{1,2}} - \frac{1}{4}}{r_{1,2}^2} - E_{n_{1,2}, \tilde{n}_1} \right\} \eta_{n_{1,2}, \tilde{n}_1}(w_{1,2}, r_{1,2}, \tilde{r}_1) = 0. \end{aligned} \quad (26)$$

For $\mu = 0$, the solution of Eq.(26) is simply given in terms of the Laguerre Polynomials L_k^ℓ and Hermite polynomials H_M [27] :

$$\begin{aligned} \eta_{M, k_{1,2}, \tilde{k}_1, n_{1,2}, \tilde{n}_1}(w_{1,2}, r_{1,2}, \tilde{r}_1) &= H_M(\sqrt{\omega} w_{1,2}) \exp(-\omega w_{1,2}^2/2) \\ & r_{1,2}^{b_{n_{1,2}}+1/2} L_{k_{1,2}}^{b_{n_{1,2}}}(\omega r_{1,2}^2) \exp(-\omega r_{1,2}^2/2) \prod_{i=1}^3 r_{i,1}^{b_{n_{i,1}}+1/2} L_{k_{i,1}}^{b_{n_{i,1}}}(\omega r_{i,1}^2) \exp(-\omega r_{i,1}^2/2) \\ M &= 0, 1, 2, \dots, \quad (\forall i), i = 1, 2, 3, k_{i,1} = 0, 1, 2, \dots, \quad k_{1,2} = 0, 1, 2, \dots, \end{aligned} \quad (27)$$

and corresponds to the energy spectrum

$$E_{M, k_{1,2}, \tilde{k}_1, n_{1,2}, \tilde{n}_1} = \omega \left(9 + 2b_{n_{1,2}} + 2 \sum_{i=1}^3 b_{n_{i,1}} + 2M + 4k_{n_{1,2}} + 4 \sum_{i=1}^3 k_{n_{i,1}} \right). \quad (28)$$

The energy (28) can be rewritten, thanks to Eqs.(11,23)

$$E_{M, k_{1,2}, \tilde{k}_1, n_{1,2}, \tilde{n}_1} = 2\omega \left\{ \frac{21}{2} + 3n_{1,2} + 3a_{1,2} + \sum_{i=1}^3 (3n_{i,1} + 3a_{i,1}) + M + 2k_{n_{1,2}} + 2 \sum_{i=1}^3 k_{n_{i,1}} \right\}. \quad (29)$$

For $\mu \neq 0$, we introduce the hyperspherical transformation [28] :

$$\begin{aligned} w_{1,2} &= r \cos \alpha, & r_{1,2} &= r \sin \alpha \cos \theta \\ 0 &\leq r < \infty & 0 &\leq \alpha \leq \pi \\ r_{1,1} &= r \sin \alpha \sin \theta \cos \beta, & r_{2,1} &= r \sin \alpha \sin \theta \sin \beta \sin \phi, & r_{3,1} &= r \sin \alpha \sin \theta \sin \beta \cos \phi, \\ 0 &\leq \theta \leq \frac{\pi}{2}, & 0 &\leq \beta \leq \frac{\pi}{2}, & 0 &\leq \phi \leq \frac{\pi}{2}. \end{aligned} \quad (30)$$

The Schrödinger equation (26) is then written as:

$$\left\{ -\frac{\partial^2}{\partial r^2} - \frac{4}{r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2} \left[-\frac{\partial^2}{\partial \alpha^2} - 3 \cot \alpha \frac{\partial}{\partial \alpha} + \frac{1}{\sin^2 \alpha} \left(-\frac{\partial^2}{\partial \theta^2} - 2 \cot \theta \frac{\partial}{\partial \theta} \right. \right. \right. \\ \left. \left. + \frac{B_{n_{1,2}} - \frac{1}{4}}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \left(-\frac{\partial^2}{\partial \beta^2} - \cot \beta \frac{\partial}{\partial \beta} + \frac{B_{n_{1,1}} - \frac{1}{4}}{\cos^2 \beta} + \frac{1}{\sin^2 \beta} \left(-\frac{\partial^2}{\partial \phi^2} \right. \right. \right. \right. \\ \left. \left. \left. + \frac{B_{n_{2,1}} - \frac{1}{4}}{\sin^2 \phi} + \frac{B_{n_{3,1}} - \frac{1}{4}}{\cos^2 \phi} \right) \right) \right] - E_{n_{1,2}, \tilde{n}_1} \right\} \eta_{n_{1,2}, \tilde{n}_1}(r, \alpha, \theta, \beta, \phi) = 0. \quad (31)$$

This Hamiltonian may be mapped to the problem of one particle in the five dimensional space with a non central potential of the form

$$V(r, \alpha, \beta, \theta, \phi) = f_1(r) + \frac{1}{r^2 \sin^2 \alpha} \left[f_2(\theta) + \frac{1}{\sin^2 \theta} \left(f_3(\beta) + \frac{f_4(\phi)}{\sin^2 \beta} \right) \right]. \quad (32)$$

The problem becomes then separable in the five variables $\{r, \alpha, \theta, \beta, \phi\}$. To find the solution we factorize the wave function as follows :

$$\eta_{k, \ell, j, m, i, n_{1,2}, \tilde{n}_1}(r, \alpha, \theta, \beta, \phi) = \frac{F_{k, \ell, j, m, i, n_{1,2}, \tilde{n}_1}(r)}{r^2} \frac{G_{\ell, j, m, i, n_{1,2}, \tilde{n}_1}(\alpha)}{\sin^{3/2} \alpha} \frac{\Theta_{j, m, i, n_{1,2}, \tilde{n}_1}(\theta)}{\sin \theta} \frac{Q_{m, i, \tilde{n}_1}(\beta)}{\sqrt{\sin \beta}} \zeta_{i, n_{2,1}, n_{3,1}}(\phi). \quad (33)$$

Accordingly, Eq.(31) separates in five decoupled differential equations:

$$\left(-\frac{d^2}{d\phi^2} + \frac{B_{n_{2,1}} - \frac{1}{4}}{\sin^2 \phi} + \frac{B_{n_{3,1}} - \frac{1}{4}}{\cos^2 \phi} \right) \zeta_{i, n_{2,1}, n_{3,1}}(\phi) = R_{i, n_{2,1}, n_{3,1}} \zeta_{i, n_{2,1}, n_{3,1}}(\phi), \quad (34)$$

$$\left(-\frac{d^2}{d\beta^2} + \frac{R_{i, n_{2,1}, n_{3,1}} - \frac{1}{4}}{\sin^2 \beta} + \frac{B_{n_{1,1}} - \frac{1}{4}}{\cos^2 \beta} \right) Q_{m, i, \tilde{n}_1}(\beta) = C_{m, i, \tilde{n}_1} Q_{m, i, \tilde{n}_1}(\beta), \quad (35)$$

$$\left(-\frac{d^2}{d\theta^2} + \frac{C_{m, i, \tilde{n}_1} - \frac{1}{4}}{\sin^2 \theta} + \frac{B_{n_{1,2}} - \frac{1}{4}}{\cos^2 \theta} \right) \Theta_{j, m, i, n_{1,2}, \tilde{n}_1}(\theta) = D_{j, m, i, n_{1,2}, \tilde{n}_1} \Theta_{j, m, i, n_{1,2}, \tilde{n}_1}(\theta), \quad (36)$$

$$\left(-\frac{d^2}{d\alpha^2} + \frac{D_{j, m, i, n_{1,2}, \tilde{n}_1} - \frac{1}{4}}{\sin^2 \alpha} \right) G_{\ell, j, m, i, n_{1,2}, \tilde{n}_1}(\alpha) = A_{\ell, j, m, i, n_{1,2}, \tilde{n}_1} G_{\ell, j, m, i, n_{1,2}, \tilde{n}_1}(\alpha), \quad (37)$$

and

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell, j, m, i, n_{1,2}, \tilde{n}_1} - \frac{1}{4}}{r^2} \right) F_{k, \ell, j, m, i, n_{1,2}, \tilde{n}_1}(r) = E_{k, \ell, j, m, i, n_{1,2}, \tilde{n}_1} F_{k, \ell, j, m, i, n_{1,2}, \tilde{n}_1}(r). \quad (38)$$

The regular solutions of (34) on the interval $]0, \pi/2[$, with Dirichlet conditions at the boundaries, are given by the expressions [7, 19]

$$\zeta_{i, n_{2,1}, n_{3,1}}(\phi) = (\sin \phi)^{b_{n_{2,1}} + \frac{1}{2}} (\cos \phi)^{b_{n_{3,1}} + \frac{1}{2}} P_i^{(b_{n_{2,1}}, b_{n_{3,1}})}(\cos 2\phi), \quad i = 0, 1, 2, \dots \quad (39)$$

in terms of the Jacobi Polynomials, and are associated to the eigenvalues

$$R_{i,n_{2,1},n_{3,1}} = r_{i,n_{2,1},n_{3,1}}^2, \quad r_{i,n_{2,1},n_{3,1}} = (2i + 1 + b_{n_{2,1}} + b_{n_{3,1}}) \quad i = 0, 1, 2, \dots \quad (40)$$

Here $b_{n_{2,1}}$ and $b_{n_{3,1}}$ are taken from equations (11,12).

The second angular equation for the polar angle β reads :

$$\left(-\frac{d^2}{d\beta^2} + \frac{r_{i,n_{2,1},n_{3,1}}^2 - \frac{1}{4}}{\sin^2 \beta} + \frac{b_{n_{1,1}}^2 - \frac{1}{4}}{\cos^2 \beta} - C_{m,i,\tilde{n}_1} \right) Q_{m,i,\tilde{n}_1}(\beta) = 0. \quad (41)$$

Its regular solutions in $]0, \pi/2[$ with Dirichlet conditions are

$$Q_{m,i,n_{1,2},\tilde{n}_1}(\beta) = (\sin \beta)^{r_{i,n_{2,1},n_{3,1}} + \frac{1}{2}} (\cos \beta)^{b_{n_{1,1}} + \frac{1}{2}} P_m^{(r_{i,n_{2,1},n_{3,1}}, b_{n_{1,1}})}(\cos 2\beta), \quad (42)$$

$$0 \leq \beta \leq \frac{\pi}{2}, \quad m = 0, 1, 2, \dots \quad (43)$$

They are associated to the eigenvalue :

$$C_{m,i,\tilde{n}_1} = c_{m,i,\tilde{n}_1}^2, \quad c_{m,i,\tilde{n}_1} = (2m + 1 + r_{i,n_{2,1},n_{3,1}} + b_{n_{1,1}}) \quad m = 0, 1, 2, \dots \quad (44)$$

Taking into account Eq.(40) we have :

$$c_{m,i,\tilde{n}_1} = (2i + 2m + 2 + b_{n_{1,1}} + b_{n_{2,1}} + b_{n_{3,1}}) \quad i = 0, 1, 2, \dots \quad m = 0, 1, 2, \dots \quad (45)$$

The regular solutions Eq.(36) in $]0, \pi/2[$ with Dirichlet conditions read, taking into account Eqs.(23, 45),

$$\Theta_{j,m,i,n_{1,2},\tilde{n}_1}(\theta) = (\sin \theta)^{c_{m,i,\tilde{n}_1} + \frac{1}{2}} (\cos \theta)^{b_{n_{1,2}} + \frac{1}{2}} P_j^{(c_{m,i,\tilde{n}_1}, b_{n_{1,2}})}(\cos 2\theta), \quad (46)$$

$$0 \leq \theta \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \dots$$

The eigenvalues $D_{j,m,i,n_{1,2},\tilde{n}_1}$ of Eq.(36) are given by

$$D_{j,m,i,n_{1,2},\tilde{n}_1} = d_{j,m,i,n_{1,2},\tilde{n}_1}^2, \quad d_{j,m,i,n_{1,2},\tilde{n}_1} = (2j + c_{m,i,\tilde{n}_1} + b_{n_{1,2}} + 1), \quad j = 0, 1, 2, \dots, \quad (47)$$

and taking into account (45)

$$d_{j,m,i,n_{1,2},\tilde{n}_1} = (2j + 2m + 2i + b_{n_{1,1}} + b_{n_{2,1}} + b_{n_{3,1}} + b_{n_{1,2}} + 3), \quad (48)$$

$$j = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots, \quad i = 0, 1, 2, \dots$$

The regular eigensolutions and corresponding eigenvalues of Eq.(37) in the interval $]0, \pi[$ read, respectively [19],

$$G_{\ell,j,m,i,n_{1,2},\tilde{n}_1}(\alpha) = (\sin \alpha)^{d_{j,m,i,n_{1,2},\tilde{n}_1} + \frac{1}{2}} C_\ell^{(d_{j,m,i,n_{1,2},\tilde{n}_1} + \frac{1}{2})}(\cos \alpha), \quad \ell = 0, 1, 2, \dots, \quad (49)$$

$$A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = a_{\ell,j,m,i,n_{1,2},\tilde{n}_1}^2, \quad (50)$$

$$a_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = \left(\ell + d_{j,m,i,n_{1,2},\tilde{n}_1} + \frac{1}{2} \right) \quad \ell = 0, 1, 2, \dots,$$

and taking into account (48)

$$a_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = \left(\ell + 2j + 2m + 2i + \sum_{M=1}^3 b_{n_{M,1}} + b_{n_{1,2}} + \frac{7}{2} \right),$$

$$\ell = 0, 1, 2, \dots, j = 0, 1, 2, \dots, m = 0, 1, 2, \dots, i = 0, 1, 2, \dots \quad (51)$$

Our choices $b_{n_{M,1}} > 0$, $M = 1, 2, 3$, and $b_{n_{1,2}} > 0$ ensure that the Hamiltonians of Eqs.(34,35,36,37) are self-adjoint operators.

Finally, the reduced radial equation reads

$$\left(-\frac{d^2}{dr^2} + \omega^2 r^2 + \frac{\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} - \frac{1}{4}}{r^2} - E_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1} \right) F_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}(r) = 0. \quad (52)$$

We introduce the auxiliary parameter $\kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1}$ defined by

$$\kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1}^2 = \mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}, \quad \kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = \sqrt{\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}}. \quad (53)$$

The solution of the radial equation (52) reads [19]

$$F_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}(r) = r^{\kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1} + \frac{1}{2}} \exp\left(-\frac{\omega r^2}{2}\right) L_k^{(\kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1})}(\omega r^2), \quad k = 0, 1, 2, \dots, \quad (54)$$

L_k^κ being the generalized Laguerre polynomials. The eigenenergies are given by

$$E_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1} = 2\omega(2k + \kappa_{\ell,j,m,i,n_{1,2},\tilde{n}_1} + 1), \quad k = 0, 1, 2, \dots \quad (55)$$

Note that the reduced radial equation (52) is nothing but the usual 3-dimensional harmonic oscillator equation, $(\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} - 1/4)/r^2$ replacing the centrifugal barrier. The square integrable solutions are well known, putting a limit on the coefficient of the $1/r^2$ term, namely $(\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}) > 0$. Note that taking $\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = 0$ leads to several self-adjoint extensions differing by a phase. This fact has been discussed in [19]. More details can be found in [29, 30]. It has to be noted that for attractive centrifugal barriers, $\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} < 0$, the problem of collapse appears, unless regularization procedures are carried out [31, 32, 33, 34].

Taking into account the definition of $A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}$, Eq.(50,51), we have

$$\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1} = \mu + \left(\ell + 2j + 2m + 2i + 3 \sum_{M=1}^3 (n_{M,1} + a_{M,1}) + 3n_{1,2} + 3a_{1,2} + \frac{19}{2} \right)^2 > 0$$

$$\forall \ell \geq 0, \forall j \geq 0, \forall m \geq 0, \forall i \geq 0, \forall n_{1,2} \geq 0, \forall n_{1,1} \geq 0, \forall n_{2,1} \geq 0, \forall n_{3,1} \geq 0 \quad (56)$$

for every positive μ .

The quantity $\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}$ is minimal for $\tilde{n}_1 = \tilde{0}$, $n_{1,2} = 0$, $i = 0$, $j = 0$, $m = 0$, $\ell = 0$ and $a_{M,1} = 0$ ($\forall M = 1, 2, 3$), $a_{1,2} = 0$ (we recall that $a_{M,1}$, $M = 1, 2, 3$, $a_{1,2} \geq 0$ see (12,24)). The positivity of $\mu + A_{\ell,j,m,i,n_{1,2},\tilde{n}_1}$ puts constraints on negative values of μ , namely

$$-\left(\frac{19}{2}\right)^2 < \mu \leq 0. \quad (57)$$

Collecting all pieces, we conclude that the physically acceptable (non normalized) solutions of the Schrödinger equation (7) are given, in a compact and symmetrized form, by

$$\begin{aligned}
\Psi_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}(r, \alpha, \theta, \beta, \phi, \varphi_{1,2}, \tilde{\varphi}_1) &= r \sqrt{\mu + (19/2 + 3a_{1,2} + 3n_{1,2} + \sum_{M=1}^3 (3a_{M,1} + 3n_{M,1}) + \ell + 2j + 2m + 2i)^2 - 7/2} \\
&\times L_k^{\sqrt{\mu + (19/2 + 3a_{1,2} + 3n_{1,2} + \sum_{M=1}^3 (3a_{M,1} + 3n_{M,1}) + \ell + 2j + 2m + 2i)^2}}(\omega r^2) \exp\left(-\frac{\omega r^2}{2}\right) \\
&\times (\sin \alpha)^{6+3a_{1,2}+3n_{1,2}+\sum_{M=1}^3 (3a_{M,1}+3n_{M,1})+2i+2j+2m} \\
&\times C_\ell^{19/2+3a_{1,2}+3n_{1,2}+\sum_{M=1}^3 (3a_{M,1}+3n_{M,1})+2i+2j+2m}(\cos \alpha) \\
&\times (\sin \theta)^{9/2+\sum_{M=1}^3 (3a_{M,1}+3n_{M,1})+2i+2m}(\cos \theta)^{3/2+3a_{1,2}+3n_{1,2}} \\
&\times P_j^{13/2+\sum_{M=1}^3 (3a_{M,1}+3n_{M,1})+2i+2m, 3/2+3a_{1,2}+3n_{1,2}}(\cos 2\theta) \\
&\times (\sin \beta)^{3+2i+3a_{2,1}+3a_{3,1}+3n_{2,1}+3n_{3,1}}(\cos \beta)^{3/2+3a_{1,1}+3n_{1,1}} P_m^{4+3a_{2,1}+3a_{3,1}+3n_{2,1}+3n_{3,1}, 3/2+3a_{1,1}+3n_{1,1}}(\cos 2\beta) \\
&\times (\sin \phi)^{3a_{2,1}+3n_{2,1}+3/2}(\cos \phi)^{3a_{3,1}+3n_{3,1}+3/2} P_i^{3/2+3a_{2,1}+3n_{2,1}, 3/2+3a_{3,1}+3n_{3,1}}(\cos 2\phi) \\
&\times |\sin 3\varphi_{1,2}|^{\frac{1}{2}+a_{1,2}} C_{n_{1,2}}^{(\frac{1}{2}+a_{1,2})}(\cos 3\varphi_{1,2}) \prod_{M=1}^3 |\sin 3\varphi_{M,1}|^{\frac{1}{2}+a_{M,1}} C_{n_{M,1}}^{(\frac{1}{2}+a_{M,1})}(\cos 3\varphi_{M,1}), \quad (58)
\end{aligned}$$

with

$$\begin{aligned}
k &= 0, 1, 2, \dots, \ell = 0, 1, 2, \dots, j = 0, 1, 2, \dots, m = 0, 1, 2, \dots, i = 0, 1, 2, \dots, \\
n_{1,2} &= 0, 1, 2, \dots, n_{1,1} = 0, 1, 2, \dots, n_{2,1} = 0, 1, 2, \dots, n_{3,1} = 0, 1, 2, \dots \\
(\forall M) \quad (1 \leq M \leq 3) \quad &0 \leq \varphi_{M,1} \leq \frac{\pi}{3}, \quad 0 \leq \varphi_{1,2} \leq \frac{\pi}{3} \quad 0 \leq \phi \leq \frac{\pi}{3}, \\
&0 \leq \beta \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \alpha \leq \pi \quad 0 \leq r \leq \infty \\
(\forall M) \quad (1 \leq M \leq 3) \quad &a_{M,1} = \frac{1}{2} \sqrt{1 + 2\lambda_{M,1}}, \quad a_{1,2} = \frac{1}{2} \sqrt{1 + 2\lambda_{1,2}}.
\end{aligned}$$

It has to be noticed that, for Bose statistics, a δ pathology occurs in (58) for $a_{M,1} = 1/2$ ($\lambda_{M,1} = 0$, $M = 1, 2, 3$) and $a_{1,2} = 1/2$ ($\lambda_{1,2} = 0$).

The normalization constant $N_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}$ of the wave function $\Psi_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}$, Eq.(58), can be calculated from

$$\begin{aligned}
&\int_0^{+\infty} r^8 dr \int_0^\pi \sin^7 \alpha d\alpha \int_0^{\pi/2} \sin^5 \theta \cos \theta d\theta \int_0^{\pi/2} \sin^3 \beta \cos \beta d\beta \int_0^{\pi/2} \sin 2\phi d\phi \int_0^{\pi/3} d\varphi_{1,2} \\
&\prod_{M=1}^3 \int_0^{\pi/3} d\varphi_{M,1} \Psi_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}(r, \alpha, \theta, \beta, \phi, \varphi_{1,2}, \tilde{\varphi}_1) \Psi_{k',\ell',j',m',i',n'_{1,2},\tilde{n}'_1}(r, \alpha, \theta, \beta, \phi, \varphi_{1,2}, \tilde{\varphi}_1) \\
&= \delta_{k,k'} \delta_{\ell,\ell'} \delta_{j,j'} \delta_{m,m'} \delta_{i,i'} \delta_{n_{1,2},n'_{1,2}} \delta_{n_{1,1},n'_{1,1}} \delta_{n_{2,1},n'_{2,1}} \delta_{n_{3,1},n'_{3,1}} N_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}. \quad (59)
\end{aligned}$$

Use is made of the orthogonality properties of Gegenbauer, Jacobi and Laguerre polynomials [27]. The normalization constant $N_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}$ can be worked out analytically. Its expression involves products of norms of Gegenbauer, Jacobi and Laguerre polynomials. The eigenenergies are :

$$\frac{E_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1}}{2\omega} = 2k+1 + \sqrt{\mu + \left(\ell + 2j + 2m + 2i + 3n_{1,2} + 3a_{1,2} + 3 \sum_{M=1}^3 (n_{M,1} + a_{M,1}) + \frac{19}{2} \right)^2} \quad (60)$$

$$k = 0, 1, 2, \dots, \ell = 0, 1, 2, \dots, j = 0, 1, 2, \dots, m = 0, 1, 2, \dots, i = 0, 1, 2, \dots,$$

$$n_{1,2} = 0, 1, 2, \dots, n_{1,1} = 0, 1, 2, \dots, n_{2,1} = 0, 1, 2, \dots, n_{3,1} = 0, 1, 2, \dots,$$

$$(\forall M) \quad (1 \leq M \leq 3) \quad a_{M,1} = \frac{1}{2} \sqrt{1 + 2\lambda_{M,1}}, \quad a_{1,2} = \frac{1}{2} \sqrt{1 + 2\lambda_{1,2}}.$$

Setting $\mu = 0$ we have,

$$E_{k,\ell,j,m,i,n_{1,2},\tilde{n}_1} = 2\omega \left\{ \frac{21}{2} + 3n_{1,2} + 3a_{1,2} + 3 \sum_{M=1}^3 (n_{M,1} + a_{M,1}) + \ell + 2k + 2j + 2m + 2i \right\}. \quad (61)$$

This expression is equivalent to Eq.(29). Indeed, the energy spectra are similar if we identify M (of Eq.29) and ℓ (of Eq.(61)). Then, setting $L = 2k + 2j + 2m + 2i$ for Eq.(61) and $L = 2k_{n_{1,2}} + 2 \sum_{i=1}^3 k_{n_{i,1}}$ for Eq.(29) we obtain identical expressions of the energies, Eqs.(29,61). The energy spectra are thus the same.

3 The general case of $N = 3^k$ particles ($k \geq 2$)

The same procedure can be generalized to $N = 3^k$, $k \geq 2$. As before, the N particles, with coordinates $x_i, i = 1, 2, \dots, 3^k$ are confined in a harmonic well. Then, the N particles are clustered in 3^{k-1} clusters of 3 particles (clusters of the first level). The ℓ^{th} , ($\ell = 1, 2, \dots, 3^{k-1}$) cluster of the first level incorporates the three particles $x_i, i = 3\ell - 2, 3\ell - 1, 3\ell$. At the next step, the "first level clusters" are clustered in "second level clusters" of 3 clusters. In each of these "second level clusters" the "first level clusters" interact via a 2-body Calogero-type of potential given in terms of their centre of mass coordinates, labeled $w_{i,1}, i = 1, 2, \dots, 3^{k-2}$. The ℓ^{th} , ($\ell = 1, 2, \dots, 3^{k-2}$) cluster of the second level incorporates the three clusters of first level positioned at $w_{i,1}, i = 3\ell - 2, 3\ell - 1, 3\ell$. This procedure is generalized by clustering further "second level clusters" for $k \geq 3$, etc... Finally, a non-translationally invariant N -body potential is added, with coupling constant μ , namely $\mu / (\sum_{\ell=1}^{3^k} x_\ell^2)$. This quantum N -body system is represented by the following Hamiltonian :

$$\begin{aligned} H = & \sum_{\ell=1}^{3^k} \left(-\frac{\partial^2}{\partial x_\ell^2} + \omega^2 x_\ell^2 \right) + \frac{\mu}{\sum_{\ell=1}^{3^k} x_\ell^2} \\ & + \sum_{\ell=1}^{3^{k-1}} \lambda_{\ell,1} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(x_i - x_j)^2} + \sum_{\ell=1}^{3^{k-2}} \lambda_{\ell,2} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,1} - w_{j,1})^2} \\ & + \sum_{\ell=1}^{3^{k-3}} \lambda_{\ell,3} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,2} - w_{j,2})^2} + \dots + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,k}}{(w_{i,k-1} - w_{j,k-1})^2}, \quad (62) \end{aligned}$$

where we have defined the centres of mass of three-body clusters, clusters of "type 1" :

$$w_{\ell,1} = \frac{x_{3\ell-2} + x_{3\ell-1} + x_{3\ell}}{\sqrt{3}} \quad \ell = 1, 2, 3, \dots, 3^{k-1} . \quad (63)$$

The x_ℓ 's are the coordinates of the particles ($\ell = 1, 2, \dots, 3^k$). The centres of mass of nine-body clusters, clusters of "type 2", read

$$w_{\ell,2} = \frac{w_{3\ell-2,1} + w_{3\ell-1,1} + w_{3\ell,1}}{\sqrt{3}} \quad \ell = 1, 2, 3, \dots, 3^{k-2} , \quad (64)$$

etc..., and the centres of mass of 3^n -body clusters, clusters of "type n", display :

$$w_{\ell,n} = \frac{w_{3\ell-2,n-1} + w_{3\ell-1,n-1} + w_{3\ell,n-1}}{\sqrt{3}} \quad \ell = 1, 2, 3, \dots, 3^{k-n}, \quad n \geq 2 . \quad (65)$$

Starting from the Hamiltonian, Eq.(62), we introduce the change of coordinates

$$\begin{aligned} u_{\ell,1} &= \frac{1}{\sqrt{2}}(x_{3\ell-2} - x_{3\ell-1}), v_{\ell,1} = \frac{1}{\sqrt{6}}(x_{3\ell-2} + x_{3\ell-1} - 2x_{3\ell}) \\ w_{\ell,1} &= \frac{1}{\sqrt{3}}(x_{3\ell-2} + x_{3\ell-1} + x_{3\ell}) , \quad \ell = 1, 2, \dots, 3^{k-1}. \end{aligned} \quad (66)$$

The transformed Hamiltonian reads:

$$\begin{aligned} H &= \sum_{\ell=1}^{3^{k-1}} \left[-\frac{\partial^2}{\partial u_{\ell,1}^2} - \frac{\partial^2}{\partial v_{\ell,1}^2} - \frac{\partial^2}{\partial w_{\ell,1}^2} + \omega^2[u_{\ell,1}^2 + v_{\ell,1}^2 + w_{\ell,1}^2] + \frac{9\lambda_{\ell,1}[u_{\ell,1}^2 + v_{\ell,1}^2]^2}{2[u_{\ell,1}^3 - 3u_{\ell,1}v_{\ell,1}^2]^2} \right] \\ &+ \sum_{\ell=1}^{3^{k-2}} \lambda_{\ell,2} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,1} - w_{j,1})^2} + \sum_{\ell=1}^{3^{k-3}} \lambda_{\ell,3} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,2} - w_{j,2})^2} \\ &+ \dots + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,k}}{(w_{i,k-1} - w_{j,k-1})^2} + \frac{\mu}{\sum_{\ell=1}^{3^{k-1}} [u_{\ell,1}^2 + v_{\ell,1}^2 + w_{\ell,1}^2]} . \end{aligned} \quad (67)$$

Then we introduce the transformation

$$\begin{aligned} u_{\ell,2} &= \frac{1}{\sqrt{2}}(w_{3\ell-2,1} - w_{3\ell-1,1}), v_{\ell,2} = \frac{1}{\sqrt{6}}(w_{3\ell-2,1} + w_{3\ell-1,1} - 2w_{3\ell,1}) \\ w_{\ell,2} &= \frac{1}{\sqrt{3}}(w_{3\ell-2,1} + w_{3\ell-1,1} + w_{3\ell,1}) , \quad \ell = 1, 2, \dots, 3^{k-2} , \end{aligned} \quad (68)$$

and obtain the Hamiltonian

$$H = \sum_{\ell=1}^{3^{k-1}} \left[-\frac{\partial^2}{\partial u_{\ell,1}^2} - \frac{\partial^2}{\partial v_{\ell,1}^2} + \omega^2[u_{\ell,1}^2 + v_{\ell,1}^2] + \frac{9\lambda_{\ell,1}[u_{\ell,1}^2 + v_{\ell,1}^2]^2}{2[u_{\ell,1}^3 - 3u_{\ell,1}v_{\ell,1}^2]^2} \right]$$

$$\begin{aligned}
& + \sum_{\ell=1}^{3^{k-2}} \left[-\frac{\partial^2}{\partial u_{\ell,2}^2} - \frac{\partial^2}{\partial v_{\ell,2}^2} - \frac{\partial^2}{\partial w_{\ell,2}^2} + \omega^2[u_{\ell,2}^2 + v_{\ell,2}^2 + w_{\ell,2}^2] + \frac{9\lambda_{\ell,2}[u_{\ell,2}^2 + v_{\ell,2}^2]^2}{2[u_{\ell,2}^3 - 3u_{\ell,2}v_{\ell,2}^2]^2} \right] \\
& + \sum_{\ell=1}^{3^{k-3}} \lambda_{\ell,3} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,2} - w_{j,2})^2} + \sum_{\ell=1}^{3^{k-4}} \lambda_{\ell,4} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,3} - w_{j,3})^2} + \dots \\
& + \sum_{1 \leq i < j \leq 3} \frac{\lambda_{1,k}}{(w_{i,k-1} - w_{j,k-1})^2} + \frac{\mu}{\sum_{\ell=1}^{3^{k-1}} [u_{\ell,1}^2 + v_{\ell,1}^2] + \sum_{\ell=1}^{3^{k-2}} [u_{\ell,2}^2 + v_{\ell,2}^2 + w_{\ell,2}^2]} .
\end{aligned}$$

The procedure of successive coordinate transformations is repeated and, at the n^{th} step, $n = 2, 3, \dots, k$, we have

$$\begin{aligned}
u_{\ell,n} &= \frac{1}{\sqrt{2}}(w_{3\ell-2,n-1} - w_{3\ell-1,n-1}), \quad v_{\ell,n} = \frac{1}{\sqrt{6}}(w_{3\ell-2,n-1} + w_{3\ell-1,n-1} - 2w_{3\ell,n-1}) \\
w_{\ell,n} &= \frac{1}{\sqrt{3}}(w_{3\ell-2,n-1} + w_{3\ell-1,n-1} + w_{3\ell,n-1}), \quad \ell = 1, 2, \dots, 3^{k-n}, \quad n = 2, 3, \dots, k,
\end{aligned}$$

and the Hamiltonian reads as :

$$\begin{aligned}
H &= \sum_{m=1}^n \left\{ \sum_{\ell=1}^{3^{k-m}} \left[-\frac{\partial^2}{\partial u_{\ell,m}^2} - \frac{\partial^2}{\partial v_{\ell,m}^2} + \omega^2[u_{\ell,m}^2 + v_{\ell,m}^2] + \frac{9\lambda_{\ell,m}[u_{\ell,m}^2 + v_{\ell,m}^2]^2}{2[u_{\ell,m}^3 - 3u_{\ell,m}v_{\ell,m}^2]^2} \right] \right\} \\
&+ \sum_{\ell=1}^{3^{k-n}} \left(-\frac{\partial^2}{\partial w_{\ell,n}^2} + \omega^2 w_{\ell,n}^2 \right) + (1 - \delta_{n,k}) \sum_{m=n+1}^k \left\{ \sum_{\ell=1}^{3^{k-m}} \lambda_{\ell,m} \sum_{3\ell-2 \leq i < j \leq 3\ell} \frac{1}{(w_{i,m-1} - w_{j,m-1})^2} \right\} \\
&+ \frac{\mu}{(\sum_{m=1}^n \sum_{\ell=1}^{3^{k-m}} [u_{\ell,m}^2 + v_{\ell,m}^2]) + \sum_{\ell=1}^{3^{k-n}} w_{\ell,n}^2} \quad (69)
\end{aligned}$$

where $\delta_{n,k}$ denotes the Kronecker symbol. The number of partial derivatives is equal to 3^k since

$$\begin{aligned}
& 2(3^{k-1} + 3^{k-2} + \dots + 3^{k-n}) + 3^{k-n} = 2 \cdot 3^{k-n}(1 + 3 + 3^2 + \dots + 3^{n-1}) + 3^{k-n} \\
& = 2 \frac{3^{k-n}(3^n - 1)}{2} + 3^{k-n} = 3^k .
\end{aligned} \quad (70)$$

At the end we have, for $n = k$:

$$\begin{aligned}
H &= \sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} \left[-\frac{\partial^2}{\partial u_{\ell,m}^2} - \frac{\partial^2}{\partial v_{\ell,m}^2} + \omega^2[u_{\ell,m}^2 + v_{\ell,m}^2] + \frac{9\lambda_{\ell,m}[u_{\ell,m}^2 + v_{\ell,m}^2]^2}{2[u_{\ell,m}^3 - 3u_{\ell,m}v_{\ell,m}^2]^2} \right] \\
&- \frac{\partial^2}{\partial w_{1,k}^2} + \omega^2 w_{1,k}^2 + \frac{\mu}{(\sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} [u_{\ell,m}^2 + v_{\ell,m}^2]) + w_{1,k}^2} . \quad (71)
\end{aligned}$$

Setting

$$\begin{aligned}
u_{\ell,m} &= r_{\ell,m} \sin \varphi_{\ell,m} \quad v_{\ell,m} = r_{\ell,m} \cos \varphi_{\ell,m}, \quad \ell = 1, 2, \dots, 3^{k-m}, \quad m = 1, 2, \dots, k, \\
0 &\leq r_{\ell,m} < \infty, \quad 0 \leq \varphi_{\ell,m} \leq 2\pi,
\end{aligned}$$

we obtain

$$\begin{aligned}
H = & \sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} \left[-\frac{\partial^2}{\partial r_{\ell,m}^2} - \frac{1}{r_{\ell,m}} \frac{\partial}{\partial r_{\ell,m}} + \omega^2 r_{\ell,m}^2 + \frac{1}{r_{\ell,m}^2} \left(-\frac{\partial^2}{\partial \varphi_{\ell,m}^2} + \frac{9\lambda_{\ell,m}}{2 \sin^2(3\varphi_{\ell,m})} \right) \right] \\
& - \frac{\partial^2}{\partial w_{1,k}^2} + \omega^2 w_{1,k}^2 + \frac{\mu}{(\sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} r_{\ell,m}^2) + w_{1,k}^2} .
\end{aligned} \tag{72}$$

Let us introduce the notation for the set of $(3^k - 1)/2$ variables

$$V_k(\mathbf{y}) \equiv \{y_{3^{k-1},1}, y_{3^{k-1}-1,1}, \dots, y_{2,1}, y_{1,1}, y_{3^{k-2},2}, \dots, y_{1,2}, \dots, y_{3,k-1}, y_{2,k-1}, y_{1,k-1}, y_{1,k}\} \tag{73}$$

For further convenience we introduce the truncated sets:

$$\hat{V}_k(\mathbf{y}) \equiv \{y_{3^{k-1}-1,1}, y_{3^{k-1}-2,1}, \dots, y_{2,1}, y_{1,1}, y_{3^{k-2},2}, \dots, y_{1,2}, \dots, y_{3,k-1}, y_{2,k-1}, y_{1,k-1}, y_{1,k}\} \tag{74}$$

$$\begin{aligned}
W_{\ell,k-m}(\mathbf{y}) \equiv & \{y_{3^{k-1},1}, y_{3^{k-1}-1,1}, \dots, y_{2,1}, y_{1,1}, y_{3^{k-2},2}, \dots, y_{1,2}, \dots, \\
& y_{3^m,k-m}, y_{3^m-1,k-m}, \dots, y_{\ell+1,k-m}, y_{\ell,k-m}\}
\end{aligned} \tag{75}$$

and

$$\begin{aligned}
\hat{W}_{\ell,k-m}(\mathbf{y}) \equiv & \{y_{3^{k-1}-1,1}, y_{3^{k-1}-2,1}, \dots, y_{2,1}, y_{1,1}, y_{3^{k-2},2}, \dots, y_{1,2}, \dots, \\
& y_{3^m,k-m}, y_{3^m-1,k-m}, \dots, y_{\ell+1,k-m}, y_{\ell,k-m}\}
\end{aligned} \tag{76}$$

We have $W_{1,k}(\mathbf{y}) \equiv V_k(\mathbf{y})$ and $\hat{W}_{1,k}(\mathbf{y}) \equiv \hat{V}_k(\mathbf{y})$, these latter both sets being defined for $k \neq 1$.

The equation (72) suggests that the wave function Ψ , solution of $H\Psi = E\Psi$, can be factorized as follows

$$\Psi(w_{1,k}, V_k(\mathbf{r}), V_k(\varphi)) = \frac{1}{\sqrt{\prod_{m=1}^k \prod_{\ell=1}^{3^{k-m}} r_{\ell,m}}} \times \chi(w_{1,k}, V_k(\mathbf{r})) \times \prod_{m=1}^k \prod_{\ell=1}^{3^{k-m}} \Phi_{(\ell,m)}(\varphi_{\ell,m}) . \tag{77}$$

This factorization permits to separate the "angular" equations from the "radial" ones. The $(3^k - 1)/2$ angular equations are solved independently:

$$\begin{aligned}
\left(-\frac{d^2}{d\varphi_{\ell,m}^2} + \frac{9\lambda_{\ell,m}}{2 \sin^2(3\varphi_{\ell,m})} \right) \Phi_{n_{\ell,m}}(\varphi_{\ell,m}) &= B_{n_{\ell,m}} \Phi_{n_{\ell,m}}(\varphi_{\ell,m}), \\
m = 1, 2, \dots, k \quad \ell = 1, 2, \dots, 3^{k-m}, &
\end{aligned} \tag{78}$$

on the interval $]0, \pi/3[$, with Dirichlet conditions at the boundaries $\varphi_{\ell,m} = 0, \pi/3$. The $B_{n_{\ell,m}}$ are the quantized eigenvalues of the equations (78) respectively given by

$$B_{n_{\ell,m}} = b_{n_{\ell,m}}^2, \quad b_{n_{\ell,m}} = 3 \left(n_{\ell,m} + \frac{1}{2} + a_{\ell,m} \right), \tag{79}$$

$$a_{\ell,m} = \frac{1}{2} \sqrt{1 + 2\lambda_{\ell,m}}, \quad n_{\ell,m} = 0, 1, 2, \dots, \quad m = 1, \dots, k, \quad \ell = 1, 2, \dots, 3^{k-m}. \tag{80}$$

The associated eigensolutions are given in terms of the Gegenbauer polynomials $C_n^{(q)}$

$$\Phi_{n_{\ell,m}}(\varphi_{\ell,m}) = (\sin 3\varphi_{\ell,m})^{\frac{1}{2}+a_{\ell,m}} C_{n_{\ell,m}}^{(\frac{1}{2}+a_{\ell,m})}(\cos 3\varphi_{\ell,m}), \quad 0 \leq \varphi_{\ell,m} \leq \frac{\pi}{3}, \quad n_{\ell,m} = 0, 1, 2, \dots \quad (81)$$

We have now to solve the following Schrödinger equation

$$\left\{ \sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} \left[-\frac{\partial^2}{\partial r_{\ell,m}^2} + \omega^2 r_{\ell,m}^2 + \frac{B_{n_{\ell,m}} - \frac{1}{4}}{r_{\ell,m}^2} - \frac{\partial^2}{\partial w_{1,k}^2} + \omega^2 w_{1,k}^2 \right] + \frac{\mu}{(\sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} r_{\ell,m}^2) + w_{1,k}^2} - E_{V_k(\mathbf{n})} \right\} \chi_{V_k(\mathbf{n})}(w_{1,k}, V_k(\mathbf{r})) = 0. \quad (82)$$

The general solution of the latter equation reads, for $\mu = 0$,

$$\begin{aligned} \chi_{n_{w_{1,k}}, V_k(\Lambda), V_k(\mathbf{n})}(w_{1,k}, V_k(\mathbf{r})) &= H_{n_{w_{1,k}}}(\sqrt{\omega} w_{1,k}) \exp(-\omega w_{1,k}^2/2) \\ &\times \prod_{m=1}^k \prod_{\ell=1}^{3^{k-m}} r_{\ell,m}^{b_{n_{\ell,m}}+1/2} L_{\Lambda_{\ell,m}}^{(b_{n_{\ell,m}})}(\omega r_{\ell,m}^2) \exp(-\omega r_{\ell,m}^2/2) \\ n_{w_{1,k}} &= 0, 1, 2, \dots, \quad \Lambda_{\ell,m} = 0, 1, 2, \dots, \quad \ell = 1, 2, \dots, 3^{k-m} \quad m = 1, 2, \dots, k \end{aligned}$$

in terms of the Hermite ($H_{n_{w_{1,k}}}$) and the Laguerre polynomials $L_{\Lambda_{\ell,m}}^{(b_{n_{\ell,m}})}$. The eigenenergy reads

$$E_{n_{w_{1,k}}, V_k(\Lambda), V_k(\mathbf{n})} = 2\omega \left\{ \frac{1}{2} + n_{w_{1,k}} + \sum_{m=1}^k \sum_{\ell=1}^{3^{k-m}} \left[1 + 2\Lambda_{\ell,m} + 3 \left(n_{\ell,m} + \frac{1}{2} + a_{\ell,m} \right) \right] \right\}. \quad (83)$$

If all the $a_{\ell,m}$'s, Eq.(80), are equal to, say, a , the summation in Eq.(83) leads to a term $a(N-1)/2$ with $N = 3^k$, contributing to the energy Eq.(83). If ($\forall \ell$) the coupling constant satisfies $\lambda_{\ell,m} = \lambda_{1,m}$ and $\lambda_{1,m} = 9^{m-1} \lambda_{1,1}$, the summation in Eq.(83), for high values of $\lambda_{1,1}$, leads to a number of terms proportional to $a_{1,1}$ equal to $3(3^{k-1} + 3^k + 3^{k+1} + 3^{2k-2}) \simeq 3^k(3^k - 1)/2 = N(N-1)/2$. Therefore we have a term like $aN(N-1)/2$ contributing to the energy.

For $\mu \neq 0$ the Hamiltonian is not separable in $\{w_{1,k}, V_k(\mathbf{r})\}$ variables. As before, we introduce hyperspherical coordinates, taking into account the fact that all components of $V_k(\mathbf{r})$ have positive values :

$$\begin{aligned} w_{1,k} &= r \cos \alpha, & 0 \leq r < \infty, & \quad 0 \leq \alpha \leq \pi \\ r_{1,k} &= r \sin \alpha \cos \beta_{1,k}, & 0 \leq \beta_{1,k} &\leq \frac{\pi}{2}, \quad k \neq 1 \\ r_{1,k-1} &= r \sin \alpha \sin \beta_{1,k} \cos \beta_{1,k-1}, & 0 \leq \beta_{1,k-1} &\leq \frac{\pi}{2} \\ r_{2,k-1} &= r \sin \alpha \sin \beta_{1,k} \sin \beta_{1,k-1} \cos \beta_{2,k-1}, & 0 \leq \beta_{2,k-1} &\leq \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned}
r_{3,k-1} &= r \sin \alpha \sin \beta_{1,k} \sin \beta_{1,k-1} \sin \beta_{2,k-1} \cos \beta_{3,k-1} , \quad 0 \leq \beta_{3,k-1} \leq \frac{\pi}{2} \\
&\dots \\
r_{1,k-m} &= \left[r \sin \alpha \prod_{j=1}^m \prod_{i=1}^{3^{j-1}} \sin \beta_{i,k-j+1} \right] \cos \beta_{1,k-m} , \quad 0 \leq \beta_{1,k-m} \leq \frac{\pi}{2} , 1 \leq m \leq k-2 \\
&\dots \\
r_{\ell,k-m} &= \left[r \sin \alpha \prod_{j=1}^m \prod_{i=1}^{3^{j-1}} \sin \beta_{i,k-j+1} \right] \times \left(\prod_{j=1}^{\ell-1} \sin \beta_{j,k-m} \right) \cos \beta_{\ell,k-m} \\
&\quad 0 \leq \beta_{\ell,k-m} \leq \frac{\pi}{2} , \quad 1 \leq \ell \leq 3^m , , \\
&\dots \\
r_{\ell,1} &= \left[r \sin \alpha \prod_{j=1}^{k-1} \prod_{i=1}^{3^{j-1}} \sin \beta_{i,k-j+1} \times \prod_{j=1}^{\ell-1} \sin \beta_{j,1} \right] \cos \beta_{\ell,1} , \quad 0 \leq \beta_{\ell,1} \leq \frac{\pi}{2} , \ell \geq 1 \\
&\dots \\
r_{3^{k-1}-1,1} &= \left[r \sin \alpha \prod_{j=1}^{k-1} \prod_{i=1}^{3^{j-1}} \sin \beta_{i,k-j+1} \times \prod_{j=1}^{3^{k-1}-2} \sin \beta_{j,1} \right] \cos \beta_{3^{k-1}-1,1} , \quad 0 \leq \beta_{3^{k-1}-1,1} \leq \frac{\pi}{2} \\
r_{3^{k-1},1} &= \left[r \sin \alpha \prod_{j=1}^{k-1} \prod_{i=1}^{3^{j-1}} \sin \beta_{i,k-j+1} \times \prod_{j=1}^{3^{k-1}-2} \sin \beta_{j,1} \right] \sin \beta_{3^{k-1}-1,1} \quad (84)
\end{aligned}$$

The Schrödinger equation (82) is then written as :

$$\begin{aligned}
&\left\{ -\frac{\partial^2}{\partial r^2} - \frac{3^k - 1}{2r} \frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu}{r^2} + \frac{1}{r^2} \left[-\frac{\partial^2}{\partial \alpha^2} - \frac{3^k - 3}{2} \cot \alpha \frac{\partial}{\partial \alpha} \right. \right. \\
&+ \frac{1}{\sin^2 \alpha} \left[-\frac{\partial^2}{\partial \beta_{1,k}^2} - \frac{3^k - 5}{2} \cot \beta_{1,k} + \frac{B_{n_{1,k}} - \frac{1}{4}}{\cos^2 \beta_{1,k}} + \right. \\
&\dots \\
&\left. \left[-\frac{\partial^2}{\partial \beta_{\ell,k-m}^2} - \frac{3^k - 3^m - 2 - 2\ell}{2} \cot \beta_{\ell,k-m} \frac{\partial}{\partial \beta_{\ell,k-m}} + \frac{B_{n_{\ell,k-m}} - \frac{1}{4}}{\cos^2 \beta_{\ell,k-m}} + \right. \right. \\
&\dots \\
&\left. \left[-\frac{\partial^2}{\partial \beta_{3^{k-1}-2,1}^2} - \cot \beta_{3^{k-1}-2,1} \frac{\partial}{\partial \beta_{3^{k-1}-2,1}} + \frac{B_{n_{3^{k-1}-2,1}} - \frac{1}{4}}{\cos^2 \beta_{1,3^{k-1}-2,1}} \right. \right. \\
&+ \frac{1}{\sin^2 \beta_{3^{k-1}-2,1}} \left[-\frac{\partial^2}{\partial \beta_{3^{k-1}-1,1}^2} + \frac{B_{n_{3^{k-1}-1,1}} - \frac{1}{4}}{\cos^2 \beta_{3^{k-1}-1,1}} + \frac{B_{n_{3^{k-1},1}} - \frac{1}{4}}{\sin^2 \beta_{3^{k-1}-1,1}} \right] \left. \right] \left. \right] \left. \right] \\
&- E_{V_k(\mathbf{n})} \} \chi_{V_k(\mathbf{n})}(r, \alpha, \hat{V}_k(\beta)) = 0 . \quad (85)
\end{aligned}$$

This equation is solved completely in the Appendix where the calculations are reported.

4 Conclusions

In this work, we have studied the exact solvability of a particular quantum system of N equal mass particles with $N = 3^k$ ($k \geq 2$), confined in an harmonic field. In this

system, the particles are clustered in clusters of 3 particles. The interaction between the particles are governed by two-body Calogero potentials inside each cluster and with several many-body potentials. The number of these potentials increases with the number of the particles. To illustrate the procedure for solving this quantum system, the particular case of 9 particles is studied ($k = 2$) and solved exactly. Namely the regular eigensolutions and the corresponding eigenenergies of the stationary Schrödinger equation are derived explicitly. The general case of $N = 3^k$ particles is also studied. Thanks to some successive appropriate coordinates transformations, the problem becomes separable, then the full solutions are explicitly derived, namely the eigen-wave functions and the energy spectrum of the corresponding Schrödinger equation. Having obtained the exact solution of this N -body particular quantum problem, with $N = 3^k (k \geq 2)$, it appears that a similar N -body problem with $N = 2^k (k \geq 2)$ particles is also exactly solvable. In this case the particles are arranged in clusters of 2-particles each, and interacting via a two-body Calogero potential inside each cluster, and with other many-body forces involved in the whole set of interactions. Other solvable N -body problems may be obtained by replacing the confining harmonic term $\sum_{i=1}^N \omega^2 x_i^2$ in the Hamiltonians considered in this paper, by an attractive "Coulomb-type" potential $-\alpha/\sqrt{\sum_{i=1}^N x_i^2}$ ($\alpha > 0$), giving rise to both bound states with negative discrete spectrum and scattering states with positive continuous spectrum.

Acknowledgements We thank Dr. R.J. Lombard for fruitful discussions. One of us (A.B.) is very grateful to the Theory Group of the IPN Orsay for its kind hospitality and to the Mentouri University of Constantine for financial support.

A Appendix

In the whole section we consider values $k \neq 1$. The Hamiltonian Eq.(85) may be mapped to the problem of one particle in the space of dimension $(3^k + 1)/2$ with a non central potential of the form

$$V(r, \alpha, \hat{V}_k(\beta)) = g(r) + \frac{1}{r^2 \sin^2 \alpha} \left[\sum_{m=0}^{k-1} \sum_{\ell=1}^{3^m} \dots (1 - \delta_{m,k-1} \delta_{\ell, 3^{k-1}}) [f_{\ell, k-m}(\beta_{\ell, k-m}) \right. \\ \left. + (1 - \delta_{m,k-1} \delta_{\ell, 3^{k-1}-1}) \frac{1}{\sin^2(\beta_{\ell, k-m})} [\dots \dots [f_{3^{k-1}-1, 1}(\beta_{3^{k-1}-1, 1})]]] \right].$$

The problem becomes then separable in the $(3^k + 1)/2$ variables $\{r, \alpha, V_k(\beta)\}$. For the sake of simplicity we introduce the set :

$$Z_{\ell, k-m}(\Lambda, \mathbf{n}) \equiv \hat{W}_{\ell, k-m}(\Lambda) \cup W_{\ell, k-m}(\mathbf{n}), \quad (86)$$

which is the union of $\hat{W}_{\ell, k-m}(\Lambda)$ and $W_{\ell, k-m}(\mathbf{n})$, and which includes $3^k - 2$ quantum numbers. More precisely we have :

$$Z_{\ell, k-m}(\Lambda, \mathbf{n}) \equiv \{\Lambda_{3^{k-1}-1, 1}, \Lambda_{3^{k-1}-2, 1}, \dots, \Lambda_{2, 1}, \Lambda_{1, 1}, \Lambda_{3^{k-2}, 2}, \dots, \Lambda_{1, 2}, \dots, \\ \Lambda_{3^m, k-m}, \Lambda_{3^m-1, k-m}, \dots, \Lambda_{\ell+1, k-m}, \Lambda_{\ell, k-m},$$

$$\begin{aligned} & n_{3^{k-1},1}, n_{3^{k-1}-2,1}, \dots, n_{2,1}, n_{1,1}, n_{3^{k-2},2}, \dots, n_{1,2}, \dots, \\ & n_{3^m,k-m}, n_{3^m-1,k-m}, \dots, n_{\ell+1,k-m}, n_{\ell,k-m} \} \end{aligned} \quad (87)$$

To find the solution we factorize the wave function as follows :

$$\begin{aligned} \chi_{n_r, n_\alpha, Z_{1,k}}(\mathbf{\Lambda}, \mathbf{n})(r, \alpha, \hat{V}_k(\beta)) &= \frac{F_{n_r, n_\alpha, Z_{1,k}}(\mathbf{\Lambda}, \mathbf{n})(r)}{r^{(3^k-1)/4}} \frac{G_{n_\alpha, Z_{1,k}}(\mathbf{\Lambda}, \mathbf{n})(\alpha)}{(\sin \alpha)^{(3^k-3)/4}} \\ &\times \prod_{m=0}^{k-1} \prod_{\ell=1}^{3^m} (1 - \delta_{m,k-1} \delta_{\ell, 3^{k-1}}) \frac{Q_{Z_{\ell,k-m}}(\mathbf{\Lambda}, \mathbf{n})(\beta_{\ell,k-m})}{(\sin \beta_{\ell,k-m})^{(3^k-3^m-2-2\ell)/4}}, \quad k \geq 1. \end{aligned} \quad (88)$$

Accordingly, equation (85) separates into $(3^k + 1)/2$ decoupled differential equations. The first one reads :

$$\begin{aligned} & \left(-\frac{\partial^2}{\partial \beta_{3^{k-1}-1,1}^2} + \frac{B_{n_{3^{k-1}-1,1}} - \frac{1}{4}}{\cos^2 \beta_{3^{k-1}-1,1}} + \frac{B_{n_{3^{k-1},1}} - \frac{1}{4}}{\sin^2 \beta_{3^{k-1}-1,1}} \right. \\ & \left. - E_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}} \right) Q_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}}(\beta_{1,3^{k-1}-1,1}) = 0. \end{aligned} \quad (89)$$

We remind that $B_{n_{\ell,k-m}}, b_{n_{\ell,k-m}}$ are given by equations (79,80). We have :

$$\begin{aligned} E_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}} &= \epsilon_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}}^2 \\ \epsilon_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}} &= 2\Lambda_{3^{k-1}-1,1} + 1 + b_{n_{3^{k-1},1}} + b_{n_{3^{k-1}-1,1}}. \end{aligned} \quad (90)$$

The eigensolution reads :

$$\begin{aligned} Q_{\Lambda_{3^{k-1}-1,1}, V_k(\mathbf{n})}(\beta_{3^{k-1}-1,1}) &= (\sin \beta_{3^{k-1}-1,1})^{\frac{1}{2}+b_{3^{k-1},1}} (\cos \beta_{3^{k-1}-1,1})^{\frac{1}{2}+b_{3^{k-1}-1,1}} \\ &\times P_{\Lambda_{3^{k-1}-1,1}}^{(b_{3^{k-1},1}, b_{3^{k-1}-1,1})}(\cos 2\beta_{3^{k-1}-1,1}). \end{aligned} \quad (91)$$

The next equation reads,

$$\begin{aligned} & \left(-\frac{\partial^2}{\partial \beta_{3^{k-1}-2,1}^2} + \frac{B_{n_{3^{k-1}-2,1}} - \frac{1}{4}}{\cos^2 \beta_{3^{k-1}-2,1}} + \frac{E_{Z_{3^{k-1}-1,1}}(\mathbf{\Lambda}, \mathbf{n}) - \frac{1}{4}}{\sin^2 \beta_{3^{k-1}-2,1}} \right. \\ & \left. - E_{Z_{3^{k-1}-2,1}}(\mathbf{\Lambda}, \mathbf{n}) \right) Q_{Z_{3^{k-1}-2,1}}(\mathbf{\Lambda}, \mathbf{n})(\beta_{3^{k-1}-2,1}) = 0, \end{aligned} \quad (92)$$

taking into account :

$$E_{\Lambda_{3^{k-1}-1,1}, n_{3^{k-1},1}, n_{3^{k-1}-1,1}} \equiv E_{Z_{3^{k-1}-1,1}}(\mathbf{\Lambda}, \mathbf{n}). \quad (93)$$

We have :

$$\begin{aligned} E_{Z_{3^{k-1}-2,1}}(\mathbf{\Lambda}, \mathbf{n}) &= \epsilon_{Z_{3^{k-1}-2,1}}^2(\mathbf{\Lambda}, \mathbf{n}) \\ \epsilon_{Z_{3^{k-1}-2,1}}(\mathbf{\Lambda}, \mathbf{n}) &= 2\Lambda_{3^{k-1}-2,1} + 2\Lambda_{3^{k-1}-1,1} + 2 + b_{n_{3^{k-1}-2,1}} + b_{n_{3^{k-1}-1,1}} + b_{n_{3^{k-1},1}}. \end{aligned}$$

The corresponding eigensolution is :

$$\begin{aligned} Q_{Z_{3^{k-1}-2,1}}(\mathbf{\Lambda}, \mathbf{n}) &= (\sin \beta_{3^{k-1}-2,1})^{\frac{1}{2}+\epsilon_{Z_{3^{k-1}-1,1}}(\mathbf{\Lambda}, \mathbf{n})} \\ &\times (\cos \beta_{3^{k-1}-2,1})^{\frac{1}{2}+b_{n_{3^{k-1}-2,1}}} \times P_{\Lambda_{3^{k-1}-1,1}}^{(\epsilon_{Z_{3^{k-1}-1,1}}(\mathbf{\Lambda}, \mathbf{n}), b_{n_{3^{k-1}-2,1}})}(\cos 2\beta_{3^{k-1}-2,1}). \end{aligned}$$

The procedure is followed, and at the m^{th} step, we obtain :

$$\left(-\frac{\partial^2}{\partial \beta_{\ell,k-m}^2} + \frac{B_{n_{\ell,k-m}} - \frac{1}{4}}{\cos^2 \beta_{\ell,k-m}} + \frac{E_{Z_{\ell+1,k-m}(\mathbf{\Lambda}, \mathbf{n})} - \frac{1}{4}}{\sin^2 \beta_{\ell,k-m}} - E_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})} \right) Q_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})}(\beta_{\ell,k-m}) = 0 . \quad (94)$$

with

$$E_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})} = \epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})}^2 . \quad (95)$$

We have

$$\begin{aligned} \epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})} &= (1 - \delta_{m,k-1}) \left(\sum_{i=m+1}^{k-1} \sum_{j=1}^{3^i} (2\Lambda_{j,k-i} + 1)(1 - \delta_{i,k-1}\delta_{j,3^i}) + b_{j,k-i} \right) \\ &+ \sum_{j=\ell}^{3^m} [(2\Lambda_{j,k-m} + 1)(1 - \delta_{m,k-1}\delta_{j,3^m}) + b_{j,k-m}] . \end{aligned} \quad (96)$$

The eigensolution reads

$$\begin{aligned} Q_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})}(\beta_{\ell,k-m}) &= (\sin \beta_{\ell,k-m})^{\epsilon_{Z_{\ell+1,k-m}(\mathbf{\Lambda}, \mathbf{n})} + \frac{1}{2}} (\cos \beta_{\ell,k-m})^{b_{\ell,k-m} + \frac{1}{2}} \\ &\times P_{\Lambda_{k-m}}^{(\epsilon_{Z_{\ell+1,k-m}(\mathbf{\Lambda}, \mathbf{n})}, b_{n_{\ell,k-m}})}(\cos 2\beta_{\ell,k-m}), \\ &0 \leq \beta_{\ell,k-m} \leq \frac{\pi}{2}, \quad \Lambda_{k-m} = 0, 1, 2, \dots . \end{aligned} \quad (97)$$

The equation concerning the angular variable $\beta_{1,k}$ is written as

$$\left(-\frac{\partial^2}{\partial \beta_{1,k}^2} + \frac{B_{n_{1,k}} - \frac{1}{4}}{\cos^2 \beta_{1,k}} + \frac{E_{Z_{2,k}(\mathbf{\Lambda}, \mathbf{n})} - \frac{1}{4}}{\sin^2 \beta_{1,k}} - E_{Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} \right) Q_{Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(\beta_{1,k}) = 0 \quad (98)$$

with

$$\epsilon_{Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} = \left(\sum_{i=0}^{k-1} \sum_{j=1}^{3^i} (2\Lambda_{j,k-i} + 1)(1 - \delta_{i,k-1}\delta_{j,3^i}) + b_{j,k-i} \right) . \quad (99)$$

The latter equation (98) includes $3^k - 2$ quantum numbers. The two last equations are :

$$\left(-\frac{\partial^2}{\partial \alpha^2} + \frac{E_{Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} - \frac{1}{4}}{\sin^2 \alpha} - E_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} \right) G_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(\alpha) = 0 \quad (100)$$

with

$$\begin{aligned} E_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} &= \epsilon_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}^2 \\ \epsilon_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} &= \left(\sum_{i=0}^{k-1} \sum_{j=1}^{3^i} (2\Lambda_{j,k-i} + 1)(1 - \delta_{i,k-1}\delta_{j,3^i}) + b_{j,k-i} \right) + n_{\alpha} + \frac{1}{2} \end{aligned} \quad (101)$$

and the radial equation

$$\left(-\frac{\partial^2}{\partial r^2} + \omega^2 r^2 + \frac{\mu + E_{n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} - \frac{1}{4}}{r^2} - E_{n_r, n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})} \right) F_{n_r, n_{\alpha}, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(r) = 0 \quad (102)$$

with

$$\frac{E_{n_r, n_\alpha, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}}{2\omega} = \left\{ 2n_r + 1 + \sqrt{\mu + \left[\left(\sum_{i=0}^{k-1} \sum_{j=1}^{3^i} (2\Lambda_{j,k-i} + 1)(1 - \delta_{i,k-1}\delta_{j,3^i}) + b_{j,k-i} \right) + n_\alpha + \frac{1}{2} \right]^2} \right\} \quad (103)$$

Setting

$$d_\alpha = \epsilon_{Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}, \quad \kappa^2 = \mu + E_{n_\alpha, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}, \quad (104)$$

and taking into account the equations (49), (54) and (97), the final solution, Eq.(88), reads :

$$\begin{aligned} \chi_{n_r, n_\alpha, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(r, \alpha, \hat{V}_k(\beta)) &= r^{\kappa - (3^k - 3)/4} L_{n_r}^\kappa(\omega r^2) \exp\left(-\frac{\omega r^2}{2}\right) \sin(\alpha)^{d_\alpha - (3^k - 5)/4} C_{n_\alpha}^{(d_\alpha + 1/2)}(\cos \alpha) \\ &\prod_{m=0}^{k-1} \prod_{\ell=1}^{3^m} (1 - \delta_{m,k-1}\delta_{\ell,3^m})(\sin \beta_{\ell,k-m})^{\epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})} + 1 - (3^k - 3^m - 2\ell)/4} (\cos \beta_{k-m})^{b_{\ell,k-m} + \frac{1}{2}} \\ &\times P_{\Lambda_{k-m}}^{(\epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})}, b_{\ell,k-m})}(\cos 2\beta_{k-m}). \end{aligned}$$

The general solution $\Psi_{n_r, n_\alpha, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(V_k(\mathbf{r}), V_k(\varphi))$, Eq.(77), in its symmetric form, reads :

$$\begin{aligned} \Psi_{n_r, n_\alpha, Z_{1,k}(\mathbf{\Lambda}, \mathbf{n})}(V_k(\mathbf{r}), V_k(\varphi)) &= r^{\kappa + 1 - 3^k/2} L_{n_r}^\kappa(\omega r^2) \exp\left(-\frac{\omega r^2}{2}\right) \sin(\alpha)^{d_\alpha + 3/2 - 3^k/2} C_{n_\alpha}^{(d_\alpha + 1/2)}(\cos \alpha) \\ &\prod_{m=0}^{k-1} \prod_{\ell=1}^{3^m} (1 - \delta_{m,k-1}\delta_{\ell,3^m})(\sin \beta_{\ell,k-m})^{\epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})} - (3^k - 3^m - 2\ell - 2)/2} (\cos \beta_{k-m})^{b_{\ell,k-m}} \\ &\times P_{\Lambda_{k-m}}^{(\epsilon_{Z_{\ell,k-m}(\mathbf{\Lambda}, \mathbf{n})}, b_{\ell,k-m})}(\cos 2\beta_{k-m}) \prod_{m=1}^k \prod_{\ell=1}^{3^{k-m}} |\sin 3\varphi_{\ell,m}|^{\frac{1}{2} + a_{\ell,m}} C_{n_{\ell,m}}^{(\frac{1}{2} + a_{\ell,m})}(\cos 3\varphi_{\ell,m}), \end{aligned} \quad (105)$$

$$\begin{aligned} (\forall \ell)(\forall m) \quad \Lambda_{\ell,k-m} &= 0, 1, 2, \dots \quad 1 \leq \ell \leq 3^{k-m}, 1 \leq m \leq k-1 \\ (\forall \ell)(\forall m) \quad n_{\ell,k-m} &= 0, 1, 2, \dots \quad 1 \leq \ell \leq 3^{k-m}, 1 \leq m \leq k-1 \\ (\forall \ell)(\forall m) \quad 0 \leq \beta_{\ell,k-m} &\leq \frac{\pi}{2}, \quad 1 \leq \ell \leq 3^{k-m}, 1 \leq m \leq k-1 \\ 0 \leq \alpha \leq \pi \quad 0 \leq r &< \infty \\ (\forall \ell)(\forall m) \quad 0 \leq \varphi_{\ell,k-m} &\leq \frac{\pi}{3}, \quad 1 \leq \ell \leq 3^{k-m}, 1 \leq m \leq k \\ (\forall \ell)(\forall m) \quad a_{\ell,m} &= \frac{1}{2} \sqrt{1 + 2\lambda_{\ell,m}}, \quad 1 \leq \ell \leq 3^{k-m}, 1 \leq m \leq k. \end{aligned} \quad (106)$$

Note that $\Lambda_{3^k,1}$ does not appear in Eqs.(106).

References

- [1] Calogero, F. : Ground State of a One-Dimensional N-Body System, *J. Math. Phys.* **10**, 2197 (1969)
- [2] Calogero, F. : Solution of the One-Dimensional N -Body-Problems with Quadratic and/or Inversely Quadratic Pair Potentials *J. Math. Phys.* **12**, 419 (1971)
- [3] Sutherland, B., Quantum ManyBody Problem in One Dimension: Ground State, *J. Math. Phys.* **12**, 246 (1971)
- [4] Sutherland, B., Exact Results for a Quantum Many-Body Problem in One Dimension, *Phys. Rev. A* **4**, 2019 (1971)
- [5] Mattis, D. C. : The many-body problem: 70 years of exactly solved quantum many-body problems. Singapore, World Scientific (1993)
- [6] Sutherland, B., Beautiful models. Singapore, World Scientific (2004)
- [7] Olshanetsky, M. A. and Perelomov, A. M. : Quantum integrable systems related to lie algebras, *Phys. Rep.* **94**, 6 (1983)
- [8] Albeverio, S., Dabrowski, L. and Fei, S-M : A remark on one-dimensional many-body problems with point interactions. *Int. J. of Mod. Phys. B.* **14**, 721 (2000)
- [9] Albeverio, S., Fei, S.-M., Kurasov, P.: On Integrability of Many-Body Problems with Point Interactions. *Operator Theory: Advances and Applications*, Vol. **132**, pp.67-76 (2002) Birkhäuser Verlag Basel/Switzerland
- [10] Calogero, F. : Solution of a Three-Body Problem in One Dimension. *J. Math. Phys.* **10**, 2191 (1969)
- [11] Calogero, F. and Marchioro, C. : Exact solution of a one-dimensional three-body scattering problem with two-body and/or three-body inverse-square potentials. *J. Math. Phys.* **15**, 1425 (1974).
- [12] Wolfes, J. : On the three body linear problem with three body interaction. *J. Math. Phys.* **15**, 1420 (1974)
- [13] Khare, A. and Bhaduri, R. K. : Some algebraically solvable three-body problems in one dimension. *J. Phys A: Math. Gen.* **27**, 2213 (1994)
- [14] Quesne, C. : Exactly solvable three-particle problem with three-body interaction. *Phys. Rev. A* **55**, 3931 (1997)
- [15] Diaf, A., Kerris, A.T., Lassaut, M. and Lombard, R.J. A new model of the Calogero type, *J. Phys. A: Math. Gen.* **39** 7305 (2006)
- [16] Fehér, L., Tsutsui, I. and Fulop, T., Inequivalent quantizations of the three-particle Calogero model constructed by separation of variables, *Nucl. Phys.* **B715** 713 (2005)

- [17] Bachkhaznadj, A., Lassaut, M., Lombard, R. J. : A model of the Calogero type in the D-dimensional space, J. Phys. A: Math. Theor. **40**8791 (2007)
- [18] Meljanac, S., Samsarov, A., Basu-Mallick, B. and Gupta, K. S. : Quantization and conformal properties of a generalized Calogero model. Eur. Phys. J. C **49**, 875 (2007)
- [19] Bachkhaznadj, A., Lassaut, M., Lombard, R. J. : A study of new solvable few body problems. J. Phys. A: Math. Theor. **42**, 065301 (2009)
- [20] Wolfes, J. : On a one-dimensional four-body scattering system. Ann. Phys. **85**, 454 (1974)
- [21] Haschke O and Rühl W, : Construction of exactly solvable quantum models of Calogero and Sutherland type with translation invariant four-particle interactions. arXiv:hep-th/9807194
- [22] Gu X Y, Ma Z Q and Sun J Q : Quantum four-body system in D dimensions. J. Math. Phys. **44** 3763 (2003)
- [23] Bachkhaznadj, A., Lassaut, M. : Extending the four-body problem of Wolfes to non-translationally invariant interactions : Few-Body Systems **54** 1945 (2013)
- [24] Bachkhaznadj, A., Lassaut, M. : Solvable Few-Body Quantum Problems : Few-Body Systems **56** 1 (2015)
- [25] Znojil, M. : Comment on "Conditionally exactly soluble class of quantum potentials", Phys. Rev. A **61**, 066101 (2000)
- [26] Reed, M. and Simon, B. : Methods of Modern Mathematical Physics vol 4. Academic, New-York (1978)
- [27] Abramowitz, M. and Stegun, I.A. : Handbook of Mathematical Functions. Dover, New York (1972)
- [28] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. : Higher Transcendental Functions vol II. McGraw-Hill, New York (1953)
- [29] Basu-Mallick, B., Ghosh, P. K. and Gupta, K. S. : Novel quantum states of the rational Calogero models without the confining interaction. Nucl. Phys. B **659**, 437 (2003)
- [30] Giri, P. R., Gupta, K. S., Meljanac, S. and Samsarov, A. : Electron capture and scaling anomaly in polar molecules. Phys. Lett. A **372**, 2967 (2008)
- [31] Case, K. M. : Singular Potentials. Phys. Rev. **80**, 797 (1950)
- [32] Gupta, K. S. and Rajeev, S. G. : Renormalization in quantum mechanics. Phys. Rev. D **48**, 5940 (1993)

- [33] Camblong, H. E., Epele, L. N., Fanchiotti, H. and Garcia Canal, C. A. : Renormalization of the Inverse Square Potential. *Phys. Rev. Lett.* **85** 1590 (2000)
- [34] Yekken, R., Lassaut, M., Lombard, R. J. : Bound States of Energy Dependent Singular Potentials. *Few-Body Systems* **54** 2113 (2013)